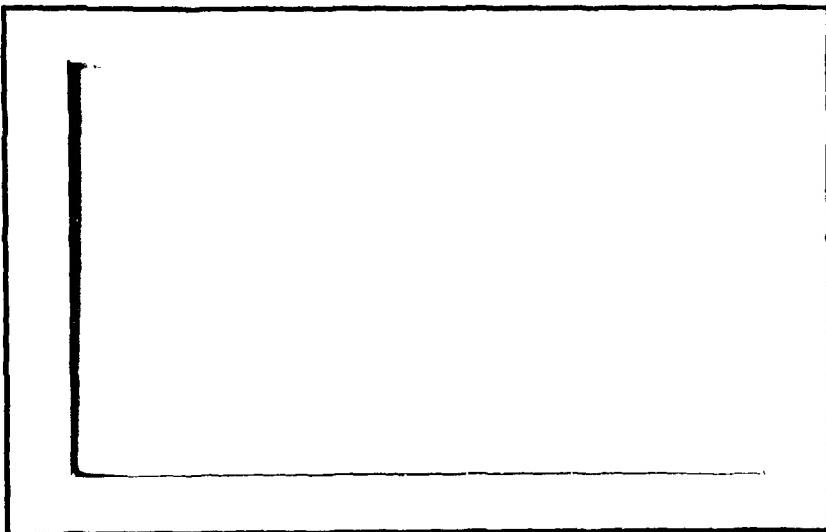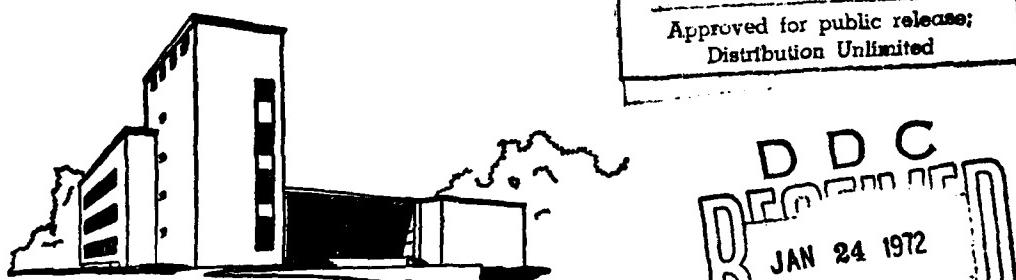# Carnegie-Mellon University

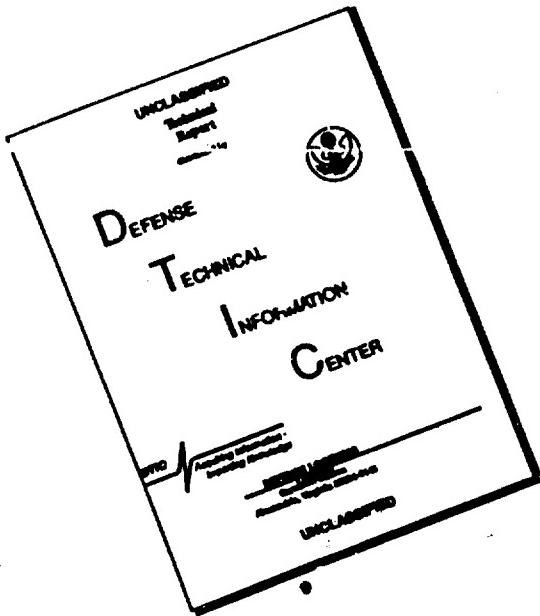PITTSBURGH, PENNSYLVANIA 15213

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19

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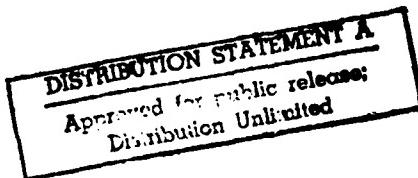
A USEFUL TRANSFORMATION OF HAMILTONIANS

OCCURRING IN OPTIMAL CONTROL PROBLEMS

IN ECONOMIC ANALYSES

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October 1971

ABSTRACT

The introduction of a discounting term into the objective functional can be troublesome in terms of analysis by the standard maximum principle formulation. This is because it renders the Hamiltonian and the adjoint equations depend explicitly on time. In finite horizon problems it makes the switching point analysis difficult. In case of infinite horizon, which is usual in economic problems, it does not admit long-run stationary equilibriums. A technique discussed by Arrow [1] to alleviate these difficulties is applied to various standard control problems occuring in economics and management science. This transforms the Hamiltonian and the corresponding adjoint system into an explicitly time-independent form and hence autonomous in all but one case. A natural consequence of the transformation is current-value interpretations of the Hamiltonian and the adjoint variables. Finally, it is noted that for the transformed systems so obtained, Hamiltonian $\mathcal{H}$ = constant, no longer provides the first integral of the resulting boundary-value problems as usual in the autonomous cases.

1. A Continuous Optimal Control Problem[1/]

Consider a system, economic or other, evolving in time. At any time t, the system is in some <u>state</u>, which can be described by a vector $x(t)$.

In an optimization problem, there is some possibility of controlling the system. Thus, at any time t, there is a vector $u(t)$ which a decision maker can choose from a given set $\Omega(t)$. The $u(t)$ are known as control variables.

The dynamics of the system is described by differential equations known as <u>state equations</u>.

(1.1) $\dot{x} = f(x,u,t)$; $x(0) = \underline{x}$

By suitable choices of controls over time, alternative histories of the process can be achieved. As is usual in economic analysis, we assume that these histories can be valued in some way. Usually this is done by assuming, at each moment t, a <u>felicity</u> function $F(x(t),u(t),t)$ and then summing these felicities over time.

Furthermore, in a finite horizon problem, since the horizon T is not the end of the world, the states at T will usually have some value. This value will be referred to as <u>scrap value</u> and will be denoted by a function $S(x(T),T)$. Now the optimal control problem is to

(1.2) $\underset{\{u(t)\varepsilon\Omega(t)\}}{\text{maximize}} \ S(x(T),T) + \int_o^T F(x(t),u(t),t)dt$

subject to (1.1).

For convenience, we also define the <u>return function</u> by

(1.3) $V(x,t_o) = \underset{u(t)}{\max} \left\{ S(x(T),T) + \int_{t_o}^T F(x(t),u(t),t)dt \right\}$.

The usual Hamiltonian $\overline{\mathcal{H}}$, in this case, is

(1.4) $\overline{\mathcal{H}} = F(x,u,t) + pf(x,u,t)$

where the vector of adjoint variables p satisfies

$$(1.5) \qquad \dot{p} = -\frac{\partial H}{\partial x} \quad ; \quad p(T) = \frac{\partial S}{\partial x(T)} \quad .$$

From the Hamilton-Jacobi Theory, we also know that

$$(1.6) \qquad p = \frac{\partial V}{\partial x}$$

Therefore, we can call p the __marginal return v__ ..r.

Since, in economics, it is customery to assume that, in some relevant sense, future felicities are discounted relative to the present. To include this feature, let $\alpha(t)$ be the __discount rate__, we can restate our problem as

$$(1.7) \qquad \underset{u \in \Omega}{\text{maximize}} \quad S(x(T),T)\alpha(T) + \int_0^T \alpha(t) \, F(x(t),u(t),t)dt$$

subject to (1.1).

The return function definition (1.3) will be correspondingly modified to be

$$(1.8) \qquad V(x,t_0) = \underset{u(t)}{\text{max}} \; [S(x(T),T)\alpha(T) + \int_{t_0}^T \alpha(t) \, F(x(t),u(t),t)dt] \quad .$$

Since the return function in (1.8) is evaluated in terms of present value at time $t = 0$, we can divide it by $\alpha(t_0)$ to obtain the __current-value return function__:

$$(1.9) \qquad W(x,t_0) = \frac{V(x,t_0)}{\alpha(t_0)}$$

The usual Hamiltonian $\overline{\mathcal{H}}$ will be

$$(1.10) \qquad \overline{\mathcal{H}} = \alpha(t) \, F(x,u,t) + pf(x,u,t)$$

where the adjoint vector or marginal returns p satisfy

$$(1.11) \qquad \dot{p} = -\frac{\partial \overline{\mathcal{H}}}{\partial x} = -\alpha(t)\frac{\partial F}{\partial x} - p\frac{\partial f}{\partial x} \; , \quad p(T) = \alpha(T)\frac{\partial S}{\partial x(T)}$$

and from the Hamilton-Jacobi theory,

$$(1.12) \qquad p = \frac{\partial V}{\partial x} \quad .$$

Since W in (1.9) is a current-value return function, we can obtain current-value marginal returns, λ, by differentiating (1.9) with respect to the state x, i.e.,

$$(1.13) \qquad \lambda = \frac{\partial W}{\partial x} = \frac{\partial V}{\partial x} \cdot \frac{1}{\alpha} = p/\alpha .$$

Substituting p from (1.13) in (1.10) and then dividing (1.10) by $\alpha(t)$, we get, what we define to be the current-value Hamiltonian $\mathcal{H}$. Thus,

$$(1.14) \qquad \mathcal{H} = \overline{\mathcal{H}}/\alpha(t) = F(x,u,t) + \lambda f(x,u,t) .$$

Since $\alpha(t)$ is positive, the controls chosen to maximize $\overline{H}$, as in Pontryagin's maximum principle, are same as those chosen to maximize $\mathcal{H}$. All we have to do now to make the transformation complete is to express the current value adjoint vector λ in terms of current-value Hamiltonian $\mathcal{H}$. From (1.11) and (1.13),

$$(1.15) \qquad \frac{d[\alpha(t)\lambda]}{dt} = \dot{\alpha}\lambda + \alpha\dot{\lambda} = - \frac{\partial[\alpha(t)\mathcal{H}]}{\partial x} ; \quad \alpha(T)\lambda(T) = \alpha(T) \frac{\partial S}{\partial x(T)}$$

On dividing through by α, we obtain

$$(1.16) \qquad \dot{\lambda} = q(t)\lambda - \frac{\partial \mathcal{H}}{\partial x} ; \quad \lambda(T) = \frac{\partial S}{\partial x(T)}$$

where,

$$(1.17) \qquad q(t) = - \dot{\alpha}(t)/\alpha(t) .$$

Note that $q(t)$ is essentially a short-term interest rate corresponding to the system of discount factors, $\alpha(t)$. The definition (1.17) can be integrated to yield the familiar form

$$(1.18) \qquad \alpha(t) = e^{-\int_0^t q(t)dt}$$

for discounting derived from a short-term interest rate varying in time.

Infinite Horizon Case

In case of infinite horizon, which is frequently the case in economic problems, the objective function is

$$(1.19) \qquad \max_{u \in \Omega} \int_0^\infty \alpha(t) \, F(x(t),u(t),t)dt$$

subject to (1.1).

Where, for (1.19) to converge for a constant felicity F, one requires

$$(1.20) \qquad \int_0^\infty \alpha(t)dt < \infty.$$

It is frequently appropriate to make an assumption of __stationarity__, i.e.

$$(1.21) \qquad \begin{aligned} F(x,u,t) &= F(x,u) \\ f(x,u,t) &= f(x,u) \\ q(t) &= r \end{aligned}$$

where the right hand sides are independent of t with r constant. It follows from its definition (1.17) and the convention $\alpha(0) = 1$, that

$$(1.22) \qquad \alpha(t) = e^{-rt}, \qquad r > 0 \quad \text{for (1.20).}$$

With (1.21), we can rewrite (1.1), (1.14), and (1.16) as

$$(1.23) \qquad \dot{x} = f(x,u), \quad x(0) = \underline{x}$$

$$(1.24) \qquad \mathcal{H} = F(x,u) + \lambda f(x,u)$$

$$(1.25) \qquad \dot{\lambda} = r\lambda - \frac{\partial \mathcal{H}}{\partial x}$$

Note that (1.25) is written without the transversality condition in (1.16).[2]

Since t does not enter explicitly into the system (1.23 - 1.25) such a system is termed __autonomous__.[3] In such cases, considerable interest is usually focused on its stationary points or __equilibria__, where all motion ceases; i.e., the values of x and λ for which $\dot{x} = 0$ and $\dot{\lambda} = 0$. This notion in economics is that of __long-run stationary equilibrium__. This is defined by the triple $(\bar{x}, \bar{\lambda}, \bar{u})$ satisfying,

$$(1.26) \quad \begin{cases} f(\bar{x},\bar{u}) = 0 \\ r\bar{\lambda} = \dfrac{\partial \mathcal{H}}{\partial \bar{x}} \\ \mathcal{H}(\bar{x},\bar{y}.\bar{\lambda}) \geq \mathcal{H}(\bar{x},u,\bar{\lambda}), \quad \forall u \epsilon \Omega \end{cases}$$

Now we will state an important sufficiency theorem without proof.[4]

A Sufficiency Theorem

Let $\{x^*(t),\ \lambda^*(t),\ u^*(t)\}$ be a Pontryagin path, (a path satisfying the maximum principle), i.e.

$$(1.27) \quad \begin{cases} \dot{x}^* = f(x^*,u^*)\ , \quad x^*(0) = \underline{x} \\ \dot{\lambda}^* = r\lambda^* - \dfrac{\partial \mathcal{H}}{\partial x^*} \\ \mathcal{H}(x^*,\ u^*,\ \lambda^*) \geq \mathcal{H}(x^*,\ u,\ \lambda^*), \quad \forall u \epsilon \Omega \end{cases}$$

Further assume,

$$(1.28) \qquad \mathcal{H}^{\circ}(x,\lambda) = \max_{u} \mathcal{H}(x,u,\lambda) \text{ is a concave function of } x \text{ for given } \lambda, \text{ and}$$

$$(1.29) \quad \begin{cases} x^*(t) \to \bar{x} \\ \lambda^*(t) \to \bar{\lambda}, \quad \bar{\lambda} \geq 0; \quad (\bar{x},\bar{\lambda},\bar{u}) \text{ from } (1.26) \\ u^*(t) \to \bar{u} \end{cases}$$

Then, the Pontryagin path $\{x^*(t),\ \lambda^*(t),\ u^*(t)\}$ is an optimal path.

2. An Infinite Horizon Discrete Optimal Control Problem[5]

For the discrete case, we will treat the following control problem.

$$(2.1) \qquad \max_{u \epsilon \Omega} \ \sum_{t=0}^{\infty} \alpha_t\, F_t(x(t),u(t))$$

subject to the difference equation

$$(2.2) \qquad \Delta x(t) = x(t+1) - x(t) = f_t(x(t),u(t)); \quad x(0) = \underline{x}$$

where, for (2.1) to converge for a constant felicity F, we require a relation analogous to (1.20) in the continuous case, i.e.,

(2.3) $\sum\limits_{t=0}^{\infty} \alpha_t < \infty$

Assume also that $\alpha_0 = 1$.

Standard Hamiltonian ($\bar{\mathcal{H}}$) Formulation

To form $\bar{\mathcal{H}}$, we adjoin a vector of present value adjoint variables
$p(t+1)$, to the state equation (2.2), and add this with the summand in
(2.1). Thus:

(2.4) $\bar{\mathcal{H}} = \alpha_t \, F_t(x,u) + p(t+1)f_t(x,u)$

where, from the theory of the/maximum principle, the vector p satisfies the
difference equation:

(2.5) $\Delta p(t) = - \dfrac{\partial \bar{\mathcal{H}}}{\partial x(t)}$.

Current-Value Hamiltonian ($\mathcal{H}$) and Current-Value Adjoint Vector (λ)

First we divide $\bar{\mathcal{H}}$ in (2.4) by α_t to obtain,

(2.6) $\mathcal{H} = \bar{\mathcal{H}}/\alpha_t = F_t(x,u) + \dfrac{p(t+1)}{\alpha_t} \, f_t(x,u)$

Now we define the current-value adjoint vector λ, as in (1.13), i.e.,

(2.7) $\lambda(t) = p(t)/\alpha_t$

With (2.7), we can rewrite $\mathcal{H}$ in (2.6) and the difference equation in (2.5)
as:

(2.8) $\mathcal{H} = F_t(x,u) + \lambda(t+1) \, f_t(x,u)\beta_t$

(2.9) $\Delta\lambda(t) = q(t)\lambda(t+1) - \dfrac{\partial \mathcal{H}}{\partial x(t)}$

where,

(2.10) $\beta_t = \dfrac{\alpha_{t+1}}{\alpha_t}$,   and

(2.11) $q(t) = - \dfrac{\alpha_{t+1} - \alpha_t}{\alpha_t} = -(\beta_t - 1)$.

It is frequently appropriate to make the important <u>stationarity</u> assumption, i.e.,

$$(2.12) \quad \begin{cases} F_t\ (x(t),u(t)) = F(x(t),u(t)) \\ f_t\ (x(t),u(t)) = f(x(t),u(t)) \\ \qquad\quad \beta_t = \beta \end{cases}$$

where the right hand sides are explicitly independent of t with β constant.

Under the stationarity assumption (2.12), $\alpha_{t+1} = \beta\alpha_t$, so that $\alpha_t = \beta^t$ (since $\alpha_o = 1$). The condition $(2\cdot3)$ becomes $\beta < 1$.

In usual economic processes with infinite horizon, the stationarity (2.12) is assumed to hold. One such case is treated in [8]. In this case,

$$(2.13) \qquad \beta = \frac{1+n}{1+r} \ , \quad n < r$$

where, n is the rate of population growth and r is the social discounting rate. Using (2.10, 2.11, 2.13), the current-value system (2.8, 2.9) for this case, assumes the autonomous form.

$$(2.14) \quad \begin{cases} \mathcal{H} = F(x,u) + \lambda(t+1)\ f(x,u)\ \dfrac{1+n}{1+r} \\[2mm] \Delta\lambda(t) = (r-n)\ \lambda(t+1) - \dfrac{\partial \mathcal{H}}{\partial x(t)} \end{cases}$$

Finally, we note that a sufficiency theorem, analogous to the one in the continuous case, holds in this case.

3. An Optimal Control Problem with a Constant Lag in Control [6/]

In this section, we shall treat an important special case of the constant lag optimal control problem, i.e., a problem in which the lag appears only in the control variable. Mathematically stated, the control problem is.

$$(3.1) \qquad \max_{u\in\Omega} \int_0^\infty \alpha(t)\ F(x(t),u(t))dt$$

subject to the differential-difference equation

(3.2) $\dot{x}(t) = f(x(t), u(t), u(t-\tau))$,

with the initial conditions $(x(0) = \underline{x}$ and $u(t) = \underline{u}(t)$ for $t\epsilon[-\tau,0)$.

Since the functions F and f are already assumed to be explicitly

independent of time t, all we need to complete the stationarity assumption,

similar to (1.21) in Section 1, is to assume:

(3.3) $\alpha(t) = e^{-rt}$, $r > 0$.

Standard Hamiltonian $(\overline{\mathcal{H}})$ Formulation

Specializing the results in Kharatishvili [6] for the problem (3.1, 3.2), we

obtain the Hamiltonian $\overline{\mathcal{H}}$ as,

(3.4) $\overline{\mathcal{H}}(x(t), x(t+\tau), p(t), p(t+\tau), u(t), u(t-\tau), u(t+\tau)) = \alpha(t) \ F(x(t),u(t))$

$+ \ p(t)f(x(t),u(t),u(t-\tau))$

$+ \ [\alpha(t)F(x(t),u(t)) + p(t)f(x(t),u(t),u(t-\tau))]_{t=t+\tau}$

where the present-value adjoint vector p(t) satisfies the differential-

difference equation,

(3.5) $\dot{p}(t) = - \dfrac{\partial \overline{\mathcal{H}}}{\partial x(t)}$

Note that the first term in the square brackets in (3.4) does not depend on

u(t), and hence it can be dropped from the Hamiltonian $\overline{\mathcal{H}}$, if desired. We

will, however, keep it to show the relationship between Kharatishvili [6] and

Budelis-Bryson [4].

Current-Value Hamiltonian $(\mathcal{H})$ and Current-Value Adjoint Vector (λ)

To get the current-value Hamiltonian $\mathcal{H}$, we divide   in (3.4) by $\alpha(t)$

as before and rewrite it as,

(3.6) $\mathcal{H}(\ ) = \dfrac{\overline{\mathcal{H}}(\ )}{\alpha(t)} = F(x(t),u(t)) + \dfrac{p(t)}{\alpha(t)} \ f(x(t),u(t),u(t-\tau))$

$\alpha(t+\tau)F(x(t+\tau),u(t+\tau)) + \dfrac{p(t+\tau)}{\alpha(t)} \ f(x(t+\tau), u(t+\tau), u(t))$.

Now, as in (1.13), we define the current-value adjoint vector $\lambda(t)$ by the relation,

(3.7) $\qquad \lambda(t) = p(t)/\alpha(t)$

With this definition of $\lambda(t)$ and the fact that $\alpha(t + \tau) = \alpha(t)\alpha(\tau)$ for the form of $\alpha(t)$ assumed in (3.3), we can rewrite the current-value Hamiltonian $\mathcal{H}$ in (3.6) as:

(3.8) $\qquad \mathcal{H}(x(t),x(t+\tau),\lambda(t),\lambda(t+\tau),u(t),u(t-\tau),u(t+\tau)) = F(x(t),u(t))$
$$+ \lambda(t)\ f\ (x(t),u(t),u(t-\tau) + \alpha(\tau)F(x(t+\tau),u(t+\tau))$$
$$+ \alpha(\tau)\ \lambda(t+\tau)\ f\ (x(t+\tau),u(t+\tau),u(t)).$$

It is a simple matter to obtain the corresponding adjoint equation by differentiating (3.7) and using (3.3, 3.5, 3.7). The equation is,

(3.9) $\qquad \dot{\lambda}(t) = r\lambda(t) - \dfrac{\partial \mathcal{H}}{\partial x(t)}$.

Once again, we have been able to transform the original non-autonomous system (3.2, 3.4, 3.5) into the autonomous system (3.2, 3.8, 3.9). This transformation, as noted before, is a useful one in analyzing the long-run stationary equilibrium of the economic system under consideration [8].

We will also state the maximum principle in this case.

Theorem: Maximum Principle:

If $x^*(t)$, $u^*(t)$, $\lambda^*(t)$ is an optimal trajectory, then it must satisfy (3.2, 3.9), and

(3.10) $\qquad \mathcal{H}(x^*(t),x^*(t+\tau),\lambda^*(t),\lambda^*(t+\tau),u^*(t),u^*(t-\tau)u^*(t+\tau)) \geq$
$$\mathcal{H}(x^*(t),x^*(t+\tau),\lambda^*(t),\lambda^*(t+\tau),u,u^*(t-\tau),u^*(t+\tau)$$
$$\forall u \in \Omega \text{ and } \forall t.$$

Budelis and Bryson [4] use calculus of variations to arrive at somewhat restricted results. They define their Hamiltonian, $\overline{\mathcal{H}}_B$, corresponding to which, the current-value Hamiltonian, $\mathcal{H}_B$, is

(3.11) $\mathcal{H}_B = F(x(t),u(t)) + \lambda_B(t) \, f(x(t),u(t),u(t-\tau))$

where

(3.12) $\dot{\lambda}_B = r \, \lambda_B(t) - \dfrac{\partial \mathcal{H}_B}{\partial x(t)}$.

Assuming the optimal control in the interior of Ω, they derive the necessary condition[7], which in the current-value form, is

(3.13) $\dfrac{\partial \mathcal{H}_B}{\partial u(t)} + \alpha(\tau) \left[\dfrac{\partial \mathcal{H}_B}{\partial u(t-\tau)} \right]_{t=t+\tau} = 0$

4. An Optimal Control Problem with Continuous Lags[8]

The problem of this section arises in ge .*ralizing the problem treated in the previous section with respect to its lag structure. Mathematically, we can state the problem as one of maximizing (3.1) subject to the integro-differential equation (also known as phase equation),

(4.1) $\dot{x} = f(x(t),u(t)) + \displaystyle\int_{-\infty}^{t} g(x(\tau),u(\tau), \tau, t)d\tau$

with the initial conditions $x(t) = \underline{x}(t)$, $\forall t \leq 0$ and $u(t) = \underline{u}(t)$, $\forall t < 0$.

We will also assume $\alpha(t)$ to be of the form (3.3).

Standard Hamiltonian $(\overline{\mathcal{H}})$ Formulation

With a slight extension of the results in Bate[2], we can define $\overline{\mathcal{H}}$ as

(4.2) $\overline{\mathcal{H}}(x(t), p(\tau \geq t), u(t)) = \alpha(t)F(x(t),u(t)) + p(t)f(x(t),u(t))$

$\qquad\qquad\qquad\qquad\qquad + \displaystyle\int_{t}^{\infty} p(\tau)g(x(t),u(t),t,\tau)d\tau$

where, as before, the adjoint vector $p(t)$ satisfies the integro-differential equation:

(4.3) $\dot{p}(t) = \dfrac{\partial \overline{\mathcal{H}}}{\partial x(t)}$

Remark: We advise the reader to compare the integrals in (4.1) and (4.2).

Current-Value Hamiltonian ($\mathcal{H}$) and Current-Value Adjoint Vector (λ)

As before, the current-value Hamiltonian $\mathcal{H}$ is obtained by dividing $\overline{\mathcal{H}}$ in (4.2) by $\alpha(t)$, i.e.,

$$(4.4) \qquad \mathcal{H}(\ ) = \frac{\overline{\mathcal{H}}(\ )}{\alpha(t)} = F(x(t),u(t)) + \frac{p(t)}{\alpha(t)} f(x(t),u(t))$$

$$+ \frac{1}{\alpha(t)} \int_t^\infty p(\tau)g(x(t),u(t),t,\tau)d\tau$$

Now we define $\lambda(t) = p(t)/\alpha(t)$ as in (3.7). With this definition and the observation that $\alpha(t) = \alpha(t-\tau)\alpha(\tau)$ for the exponential form of $\alpha(\tau)$ as assumed in (3.3), we can rewrite the current-value Hamiltonian $\mathcal{H}$ in (4.4) in terms of the variables $x(t)$, $\lambda(\tau \geq t)$, and $u(t)$, i.e.,

$$(4.5) \qquad \mathcal{H}(x(t),\lambda(\tau \geq t),u(t)) = F(x(t),u(t)) + \lambda(t) f(x(t),u(t))$$

$$+ \int_t^\infty \alpha(\tau-t)\lambda(\tau)g(x(t),u(t),t,\tau)d\tau.$$

Now we can differentiate the defining relation for $\lambda(t)$ and then using (3.3, 4.3, 4.4) we can obtain the corresponding adjoint equation

$$(4.6) \qquad \dot{\lambda}(t) = r\lambda(t) - \frac{\partial \mathcal{H}}{\partial x(t)}$$

For this case, we will expand $\frac{\partial \mathcal{H}}{\partial x(t)}$ to see that the adjoint equation for $\lambda(t)$ is an integro-differential equation with leads as opposed to the state equation in (4.1) which is an integro-differential equation with lags. In its expanded form, the adjoint equation (4.6) is,

$$(4.7) \quad \dot{\lambda}(t) = [r - f_x(x(t),u(t))]\lambda(t) - F_x(x(t),u(t))$$

$$- \int_t^\infty \alpha(\tau-t)\lambda(\tau)g_x(x(t),u(t), t, \tau)d\tau,$$

where the subscript x denotes partial differentiation with respect to x. Unlike the first three sections, the system (4.1, 4.5, 4.6) is not autonomous in general. We must mention, however, that in the economic model developed in [9]

has function g explicitly independent of x. This will render the

adjoint systems (4.7) explicitly independent of time. Furthermore, in [9],

the Hamiltonian $\mathcal{H}$ and the equation (4.1) depends on time t in such a way

that it permits us to prove the existence and uniqueness of the long-run

stationary equilibrium.

Finally, a slight extension of Bate [2] of this section, which can

be stated in current-value form as the following theorem:

Theorem: Maximum Principle: If F, f, g are differentiable with respect

to x, and if $u^*(\cdot)$ is an optimal control for the system (3.1, 4.1), with

$x^*(\cdot)$ as the corresponding trajectory, then it is necessary that

    i)   There exist a nonzero vector function of time $\lambda^*(\cdot)$ satisfying

        (4.6) with $x(\cdot) = x^*(\cdot)$ and $u(\cdot) = u^*(\cdot)$.

    ii) The optimal control $u^*(t)$ satisfies

$$\mathcal{H}(x^*(t), \lambda^*(\tau \geq t), u^*(t)) \geq \mathcal{H}(x^*(t), \lambda^*(\tau \geq t), u) \qquad \forall u \in \Omega.$$

5. Concluding Remarks.

Except for the continuous lag case in section 4, the current-value

transformation renders the system explicitly time-independent and hence

autonomous, under the stationarity assumption. Such systems allow long-run

stationary equilibriums, a concept which is extremely important in economic

analyses.

A natural consequence of the transformation is the current-value interpre-

tations of the transformed adjoint variables λ. That is, $\lambda(t_0)$ is a

vector of marginal returns associated with the phase vector $x(t_0)$ when the

returns are assumed to be discounted to time $t = t_0$ and not to time $t = 0$ as

would be the case for the standard adjoint vector $p(t_0)$. Discounting the

returns to time $t = t_0$ is more natural since this is the beginning of the

range of interest at that time.

We also note that given the current-value Hamiltonian $\mathcal{H}$, the
relation which the current-value adjoint vector λ must satisfy, has
the same form in all cases, i.e.,

$$(5.1) \qquad \dot{\lambda}(t) = q(t)\lambda(t) - \frac{\partial \mathcal{H}}{\partial x(t)}$$

where $q(t) = -\dot{\alpha}(t)/\alpha(t)$.

It is very important to emphasize the fact that the form of the equation
for the current-value adjoint vector λ in relation to the current-value
Hamiltonian $\mathcal{H}$ is different from the relationship between the form of the
equation for the standard adjoint vector p and the standard Hamiltonian $\overline{\mathcal{H}}$.

The impact of this difference can be seen if we take the time derivative
of the current value Hamiltonian $\mathcal{H}$,

$$(5.2) \qquad \frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{H}}{\partial x}\dot{x} + \frac{\partial \mathcal{H}}{\partial \lambda}\dot{\lambda} + \frac{\partial \mathcal{H}}{\partial u}\frac{du}{dt} .$$

Since $\mathcal{H}$ is explicitly independent of time, the first term is zero.
Since, either $\frac{\partial \mathcal{H}}{\partial u} = 0$ or $\frac{du}{dt} = 0$ on the optimal path, the last term vanishes
on this path. Therefore, we have

$$(5.3) \qquad \frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial x}\dot{x} + \frac{\partial \mathcal{H}}{\partial \lambda}\dot{\lambda} .$$

Furthermore, the canonical form of the state equation remains unchanged
because $\dot{x} = \frac{\partial \overline{\mathcal{H}}}{\partial p} = \frac{\partial \mathcal{H}}{\partial \lambda}$. This together with (5.1) reduces $\frac{d\mathcal{H}}{dt}$ in (5.3) as

$$(5.4) \qquad \frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial x}\dot{x} + \dot{x}[q\lambda - \frac{\partial \mathcal{H}}{\partial x}]$$

$$= \dot{x} q \lambda$$

$$\neq 0$$

Notice, that if $\overline{\mathcal{H}}$ were independent of time, i.e., if original system
were autonomous, $\frac{d\mathcal{H}}{dt} = 0$. This meant that $\mathcal{H} =$ constant would have provided
the first integral of the resulting boundary value problem. Not so (as noted
before in Footnote 3), however, in the transformed autonomous system.

FOOTNOTES

1. This entire section is based on [1, Chap. 2]. For economic problems of this type; see [1,5].

2. For an excellent discussion of transversality conditions in infinite horizon cases, refer to [1, Chap. 2].

3. It should be warned that on account of the nonstandard form of (1.25) in the autonomous system (1.23 - 1.25), $\mathcal{H}$ = constant no longer provides the first integral of the two-point boundary-value problem (1.1 , 1.16) in the finite horizon case and the problem (1.23, 1.25) in infinite horizon case.

4. For a partial proof of this theorem, see [1, Chap. 2].

5. This type of problem in economics are either the discrete counterpart of the problem in Sec. 1 or they are discrete approximation of complex problems involving lags [7,9].

6. This type of problems frequently occur in economic theory. The usual examples are models involving labor where it takes a fixed amount of time, τ, to train a labor. Such models are treated in [3,8].

7. To put Budelis-Bryson [4] results in relation to Kharatishvili [6], we note

    i) $\mathcal{H} = \mathcal{H}_B + \alpha(\tau)\,\mathcal{H}_B]_{t=t+\tau}$ cf. (3.8 and 3.11)

    ii) $\dot{\lambda} = \dot{\lambda}_B$ cf. (3.9 and 3.12)

    iii) $\dfrac{\partial \mathcal{H}}{\partial u(t)} = \dfrac{\partial \mathcal{H}_B}{\partial u(t)} + \alpha(\tau)\dfrac{\partial \mathcal{H}_B}{\partial u(t-\tau)} \Big]_{t=t+\tau} = 0$ cf. (3.10 and 3.13)

8. This is the most general problem treated in this paper. The existence theory for this problem is still incomplete. Necessary conditions have been obtained by Bates [2] for a problem of slightly less general form.

In economics, this problem provides a more general and more realistic framework than the constant lag problem located in the previous section. In [9], we apply this theory to a heterogeneous labor model. Several other economic applications are possible, however, the only one reported in the economic literature is in [9].

15a

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